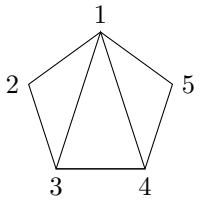
MATH2315 Linear Algebra I
Final Mock

December 2026

There are 5 questions in this mock. You must complete all questions for full marks. This mock is worth 50 marks.
Duration: 2 HOURS

Section 1: Vector Spaces

1. Consider the following *graph* of a pentagon, a collection of vertices, $V = \{1, 2, 3, 4, 5\}$, and edges represented by their endpoints, $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (3, 4), (4, 5)\}$. Consider the power sets $\mathcal{P}(V)$ and $\mathcal{P}(E)$ of V and E respectively. Define the following



addition and scaling operations:

$$\begin{aligned} P + Q &= (P \cup Q) \setminus (P \cap Q), \\ 0 \cdot P &= \emptyset, \\ 1 \cdot P &= P, \end{aligned}$$

for all P and Q in $\mathcal{P}(V)$ or $\mathcal{P}(E)$. Note that we do not define addition of $P \in \mathcal{P}(V)$ and $Q \in \mathcal{P}(E)$. The scaling is performed over the field $\mathbb{Z}_2$. One can show that both $\mathcal{P}(V)$ and $\mathcal{P}(E)$ are vector spaces with these operations.

(a) Find a basis B for $\mathcal{P}(V)$. Prove that $\text{Span}(B) = \mathcal{P}(V)$ and that B is linearly independent. (5)

(b) Construct the dual basis B^*. (3)

(c) Determine, without calculation, the dimension of $\mathcal{P}(E)$. (2)

Section 2: Linear Maps

2. Let V be a vector subspace of a finite dimensional vector space W and assume (10)

$$\dim(V) = \dim(W).$$

Prove that $V = W$.

3. Consider the map

$$L : \mathbb{R}^3 \to \mathbb{R}^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 3x - y \\ y \\ x - 2y + z \end{pmatrix}.$$

(a) Prove that L is a linear map. (2)

(b) Determine the rank and nullity of L. (4)

(c) Construct a basis for the kernel and image of L. (4)

Section 3: Eigenspaces

4. Consider the matrix

$$A = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 4 & -5 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Diagonalise A by finding a matrix P such that PAP^{-1} is diagonal. (7)

(b) Decide whether the matrix (3)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

can be diagonalised. Justify your answer.

Section 4: Inner Products

5. Consider $\mathbb{R}^3$ with the Euclidean inner product. Apply the *Gram-Schmidt orthogonalisation process* to the column vectors of

$$A = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -1 & 2 \\ 3 & 2 & -2 \end{pmatrix}$$

to find an orthonormal basis of $\mathbb{R}^3$.